

G_2 Domain Walls in M-theory

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Abstract

M-theory is considered in its low-energy limit on a G_2 manifold with non-vanishing flux. Using the Killing spinor equations for linear flux, an explicit set of first-order bosonic equations for supersymmetric solutions is found. These solutions describe a warped product of a domain wall in four-dimensional space-time and a deformed G_2 manifold. It is shown how these domain walls arise from the perspective of the associated four-dimensional $N = 1$ effective supergravity theories. We also discuss the inclusion of membrane and M5-brane sources.

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1 Introduction

It is well-known that M-theory compactified on manifolds with G_2 holonomy leads to four-dimensional effective supergravities with $N = 1$ supersymmetry [1, 2, 3]. While such compactifications on smooth G_2 spaces give rise to non-realistic theories in four dimensions, simply consisting of Abelian vector multiplets and uncharged chiral multiplets, it has been more recently discovered that phenomenologically interesting theories can arise when the G_2 space develops singularities [4, 5, 6]. Accordingly, there has been considerable activity on the subject [7]–[21] of M-theory on G_2 spaces.

As most low-energy theories from string- or M-theory, the four-dimensional effective theories from M-theory on G_2 spaces contain a number of moduli fields, associated to the possible deformations of the G_2 space. These moduli need to be fixed by a low-energy potential to have a stable vacuum and any hope for “realistic” low-energy physics. It has been known for some time [22, 23, 24] and has recently been studied in more detail [17, 25, 26, 27], that flux of anti-symmetric tensor fields is an effective tool to fix moduli. Hence, G_2 flux compactifications constitute an interesting framework for low-energy physics from M-theory.

It is these compactifications we wish to study in the present paper. Generally, there are two somewhat complementary ways to approach flux compactifications [28, 29]. Firstly, it can be studied using the higher-dimensional theory by computing the (supersymmetric) deformations of the G_2 background due to non-vanishing flux. Typically one expects the flux to deform the G_2 space, introduce warping and modify the external four-dimensional Minkowski space to a domain wall, as in the analogous case for Calabi-Yau manifolds [30, 28, 29]. Examples of these domain wall solutions have been studied in Refs. [32, 33]. A systematic analysis of such flux backgrounds can be carried out by applying the formalism of G structures to M-theory compactifications [34]–[40].

Alternatively, the problem can be approached from the viewpoint of the four-dimensional effective supergravities which arise from a flux-compactification on (un-deformed) G_2 spaces. The general structure of such theories, including a formula for the flux superpotential, has been derived in Ref. [10]. Due to the presence of the non-trivial superpotential, the simplest solution to these theories is not four-dimensional Minkowski space but, rather, a domain wall.

The main goal of this paper is analyze G_2 flux compactifications from both viewpoints and discuss the relation between them. On the one hand, we will compute the supersymmetric deformation of the 11-dimensional G_2 background due to flux. This will be done to linear order in flux, following the logic of the calculation in Ref. [30, 31]. We will then consider the associated four-dimensional $N = 1$ supergravities and find their exact BPS domain wall solutions. It is shown that these four-dimensional BPS domain walls can be viewed as the zero-mode part of the full 11-dimensional solution. We also demonstrate that the solutions can be supported by either a membrane, located entirely in the external space, or an M5-brane wrapping a three-cycle within the G_2 space.

We consider these results to be “physically” relevant in two ways. Although, the G_2 domain walls do not respect four-dimensional Poincaré invariance they may still provide a basis for phenomenologically viable compactifications. This is because non-perturbative effects which can be included in the four-dimensional effective theory may yet produce a minimum of the potential [22, 25, 41, 42] and modify the domain wall to a four-dimensional maximally symmetric space. More directly, our solutions represent the simplest way in which a membrane (or a wrapped M5-brane) would appear in “our” four-dimensional universe if it indeed arises from G_2 compactification of M-theory. In this sense, our results may provide the starting point for an analysis of topological defects in an M-theory universe.

The plan of the paper is as follows. The next section sets up some basic equations and conventions and in Section 3 we derive the first-order differential equations describing the G_2 domain walls. In Section 4, these equations are solved explicitly in terms of a mode expansion on the G_2 space. The inclusion of membrane and M5-brane sources is discussed in Section 5. In Section 6 we review the four-dimensional $N = 1$ supergravities from M-theory on G_2 spaces with flux and find their domain wall solutions. These solutions are then compared to their 11-dimensional counterparts. We conclude in Section 7. Some technical information on gamma matrix conventions and on G_2 group properties is collected in two appendices.

2 Supergravity in 11 dimensions

In this section, we summarize our conventions for 11-dimensional supergravity, following Ref. [43], and set up some notation. The bosonic part of the action is given by

$$S_{11} = \frac{1}{2} \int_{X_{11}} d^{11}x \left[\sqrt{-g} R - \frac{1}{2} F \wedge *F - \frac{1}{6} A \wedge F \wedge F \right], \quad (1)$$

where g is the 11-dimensional metric, R is the associated Ricci scalar and A is a three-form field with field strength $F = dA$. For simplicity, we have set the 11-dimensional Newton constant κ_{11} to one. Throughout our calculations we take the expectation value of the gravitino Ψ to vanish. This means that we only need concern ourselves with the bosonic equations of motion for this system, which are

$$dF = 0, \quad (2)$$

$$d^*F = -\frac{1}{2} F \wedge F, \quad (3)$$

$$R_{IJ} = \frac{1}{12} (F_{IKLM} F_J{}^{KLM} - \frac{1}{12} g_{IJ} F \cdot F). \quad (4)$$

Here $I, J \dots$ are 11-dimensional space-time indices and R_{IJ} is the Ricci tensor. For p -forms ζ, χ , the notation $\zeta \cdot \chi$ stands for the contraction of indices $\zeta_{i_1 \dots i_p} \chi^{i_1 \dots i_p}$.

The supersymmetric transformation of the gravitino is given by

$$\delta_\eta \Psi_I = D_I \eta + \frac{1}{288} F_{JKLM} (\Gamma_I{}^{JKLM} - 8 \delta_I^J \Gamma^{KLM}) \eta, \quad (5)$$

where η is an 11-dimensional Majorana spinor and D_I is the spinor covariant derivative defined by

$$D_I = \partial_I + \frac{1}{4} \omega_I{}^{\underline{IJ}} \Gamma_{\underline{IJ}}. \quad (6)$$

Underlining denotes tangent space indices, while multi-indexed symbols $\Gamma^{I_1 \dots I_p}$ denote anti-symmetrized products of gamma-matrices, as usual. Our conventions for gamma-matrices are explained in Appendix A. It can be shown [44, 45] that a solution to the Killing spinor equation $\delta_\eta \Psi_I = 0$ which also satisfies the form-field equation of motion (3) and the Bianchi identity (2) provides a solution to the Einstein equation (4) as well.

3 Finding supersymmetric G_2 domain wall solutions

3.1 General considerations

In the absence of flux, the general M-theory backgrounds which lead to four-dimensional $N = 1$ supersymmetry consist of a direct product of a G_2 manifold and four-dimensional Minkowski space.

The main goal of this paper is to understand how these backgrounds are modified in the presence of flux. As is well-known [10], flux leads to a non-vanishing moduli superpotential in the associated four-dimensional effective theory. The “simplest” solution of this theory is then a domain wall [46] rather than four-dimensional Minkowski space. We will return to this four-dimensional viewpoint later. For the 11-dimensional Ansatz in the presence of flux, this observation suggests we should accordingly modify its four-dimensional Minkowski space part to a domain wall. As we will see, also the metric on the G_2 space requires a correction due to flux.

In practice, we will work with the Killing spinor equations, the equation of motion (3) for F and its Bianchi identity (2). To simplify the problem, the flux is regarded as an expansion parameter and we will determine the flux-induced corrections to linear order. The logic of the calculation is somewhat similar to one in Ref. [30, 31] where flux corrections to Calabi-Yau backgrounds have been determined. The main result of this section will be a set of first order bosonic differential equations for these linearized corrections.

3.2 Covariantly constant spinors

As noted above, to obtain supersymmetric solutions we impose that the variation of the gravitino (5) should vanish. In the zero-flux regime, where we just consider the direct product of Minkowski space with a G_2 manifold, $M_4 \times X_7$, this amounts simply to imposing that there should be a covariantly constant spinor, η_0 , obeying

$$D_I \eta_0 = \partial_I \eta_0 + \frac{1}{4} \omega_I^{IJ} \Gamma_{IJ} \eta_0 = 0. \quad (7)$$

In Appendix B.3, we explain why G_2 manifolds will in general admit such a spinor. When we introduce flux into the equation (5), however, the condition on the Killing spinor becomes more complicated. This will lead us to perturb both the metric and spinor in order to preserve supersymmetry.

3.3 Metric Ansatz

Following the earlier discussion, we shall consider solutions to M-theory with line element corresponding to a warped product of an internal seven-dimensional space and a domain wall in four dimensions, that is,

$$ds^2 = e^{2\alpha} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2\beta} dy^2 + g_{AB} dx^A dx^B. \quad (8)$$

Here, we use indices $\mu, \nu \dots = 0, 1, 2$, and $A, B \dots = 4 \dots 10$. The three-dimensional part of the metric corresponds to the domain wall worldvolume X_3 spanned by coordinates x^μ and y is a coordinate transverse to the wall. The seven-dimensional internal space X_7 with coordinates x^A has a metric g_{AB} . Along with the warp factors α and β it generally depends on x^A and y but not on x^μ . Therefore, we have preserved three-dimensional Poincaré invariance on the domain wall worldvolume, a general requirement which we will later use to constrain the flux.

As we have explained, we would like to find a solution with metric of the above form expanding to linear order in the flux. We should, hence, think of the metric (8) as a linear perturbation of a direct product of Minkowski space with a G_2 space with Ricci-flat metric. To this end, we expand to linear order in the warp factors α and β and write the internal seven-dimensional metric as $g_{AB} = \Omega_{AB} + h_{AB}$, where Ω is a Ricci-flat metric on a G_2 space and h is the perturbation. The metric (8) then takes the form

$$ds^2 = (1 + 2\alpha) \eta_{\mu\nu} dx^\mu dx^\nu + (1 + 2\beta) dy^2 + (\Omega_{AB} + h_{AB}) dx^A dx^B \quad (9)$$

Note that to zeroth order—that is setting α , β and h to zero, and in the absence of flux—this metric indeed provides a supersymmetric solution to M-theory for the reasons given in Section §3.2. When we perturb the metric $g \mapsto g + h$, for linear h , we also perturb the spinor covariant derivative [30] as

$$D^I \mapsto D^I - \frac{1}{8} (\nabla_J h^I_K - \nabla_K h^I_J) \Gamma^{JK} . \quad (10)$$

Hence, if we want Eq. (5) to hold in the presence of flux, we should think of the corrections α , β and h as being “sourced” by flux. Our goal will be to determine their explicit form as a function of the flux, such that the corrected solution continues to preserve some supersymmetry.

3.4 Conditions on the flux

We will now write down the general form of the flux and the constraints imposed on it by the F equation of motion and the Bianchi identity.

Given we are asking for Poincaré invariance on the domain wall worldvolume X_3 , we are left with the following non-trivial components of F :

$$\begin{aligned} F_{ABCD} &= G_{ABCD} & F_{yABC} &= J_{ABC} \\ F_{A\mu\nu\rho} &= V_A \varepsilon_{\mu\nu\rho} & F_{y\mu\nu\rho} &= K \varepsilon_{\mu\nu\rho} . \end{aligned} \quad (11)$$

Note that G , J , V , K can be viewed as forms of various degree on the internal space X_7 .

Within the context of our expansion scheme, we consider flux as being first order. At linear order, we can, therefore, neglect the $F \wedge F$ term in the equation of motion (3) and work with the zeroth order metric. The F equation of motion and the Bianchi identity then simply state that

$$dF = d^* F = 0 , \quad (12)$$

where the Hodge star is with respect to the zeroth order metric. Inserting the various component (11) into Eq. (12) we find from the Bianchi identity

$$dG = dJ - G' = dV = dK - V' = 0 , \quad (13)$$

and from the equation of motion

$$d^* G - J' = d^* J = d^* V - K' = d^* K = 0 . \quad (14)$$

Here a prime denotes differentiation with respect to y and the operators d, d^* are now taken with respect to the internal space X_7 with Ricci-flat metric Ω . To summarize, the most general flux is described by a four-form G , a three-form J , a one-form V and a scalar K on the internal space X_7 which are subject to the equations (13) and (14).

3.5 Spinor Ansatz

A somewhat delicate point in computations of the Killing spinor equations is to find the most general Ansatz for the supersymmetry spinor η . Sometimes, solutions to the Killing spinor equations can be missed if, for example, a simple product Ansatz for η is used. We will, therefore, spend some time discussing this Ansatz for the spinor and finding its most general structure. All relevant conventions for spinors and gamma matrices in the various dimensions involved are collected in Appendix A.

The first point to note is that η must be Majorana, that is

$$\eta^c = \eta . \quad (15)$$

The conjugation is defined in Appendix A. In general any such spinor can be written in terms of a Dirac spinor ψ like

$$\eta = \psi + \psi^c . \quad (16)$$

If a pair of projectors P_+, P_- can be found such that

$$(P_\pm)^2 = P_\pm \quad (P_\pm \psi)^c = P_\mp \psi^c \quad P_+ + P_- = \mathbb{1}_{32} , \quad (17)$$

then we may further write

$$\begin{aligned} \eta &= \psi + \psi^c \\ &= P_+(\psi + \psi^c) + P_-(\psi + \psi^c) \\ &= (P_+\psi + (P_-\psi)^c) + ((P_+\psi)^c + P_-\psi) \\ &=: \zeta + \zeta^c \end{aligned} \quad (18)$$

where $\zeta = P_+\zeta$. Normally, P_\pm would project onto positive and negative chiralities, but there is no chirality operator in 11 dimensions so we do not yet have a physical interpretation of the manipulation above. However, when we decompose the spinor ζ as

$$\zeta = \xi_+ \otimes \chi , \quad (19)$$

where ξ_+ and χ are 4- and 7-dimensional spinors respectively, we can define a sensible pair of projectors by

$$P_\pm := \frac{1}{2} (1 \pm \gamma) \otimes \mathbb{1}_8 . \quad (20)$$

This amounts to imposing that ξ_+ is a positive chirality Weyl spinor, that is, $\xi_+ = \gamma \xi_+$. Its charge conjugate $\xi_- := \xi^c$ is then a negative chirality spinor satisfying $\xi_- = -\gamma \xi_-$. It is possible to show that, for our conventions, we have

$$\gamma^y \xi_+ = -(\xi_-)^* \quad \gamma^y \xi_- = (\xi_+)^* . \quad (21)$$

For an arbitrary complex number z we have $z^* = e^{-2i \arg(z)} z$. Of course, such a result will not in general hold for a multi-component complex object like ξ_+ . However, it turns out, after solving the Killing spinor equations, there is no loss of generality in assuming that ξ_+ indeed does satisfy such a relation. Consequently, we introduce a parameter θ such that

$$\gamma^y \xi_\pm = e^{\pm i\theta} \xi_\mp . \quad (22)$$

Note that the internal part χ of the spinor remains unconstrained by the projection, and at zeroth order in the flux will simply be the covariantly constant spinor on the G_2 manifold, χ_0 .

We now consider the perturbation of the spinor to linear order. For the 4-dimensional spinor we introduce a complex parameter ϵ such that

$$\xi = (1 + \epsilon) \xi_+ \quad (23)$$

is the first order 4-dimensional spinor. Similarly to the argument about θ above, this is not the most general perturbation of a multi-component complex object, but there will be no loss of generality later if we take ξ to be of the above form. We now use the results of Appendix B.3, to see that the most general linear perturbation of the 7-dimensional spinor χ_0 is given by

$$\chi = (1 - v_0) \chi_0 + v^A \chi_A . \quad (24)$$

Here v_0, v^A are complex variables parameterizing χ . In the full 11-dimensional picture, note that ϵ and v_0 are not really independent degrees of freedom since we can absorb ϵ into a new parameter $v_\epsilon := \epsilon - v_0$. We note here that this parameter encodes information about the variation of θ , since basic manipulation of the first order spinor gives

$$\partial\theta = 2\text{Im}(\partial v_\epsilon) . \quad (25)$$

Because our conventions for the Dirac matrices allow us to express 11-dimensional charge conjugation as

$$\zeta^c = \xi^c \otimes \chi^c , \quad (26)$$

our final Ansatz for η is then

$$\eta = \xi_+ \otimes ((1 + v_\epsilon)\chi_0 + v^A\chi_A) + \xi_- \otimes ((1 + v_\epsilon^*)\chi_0^c + (v^A)^*\chi_A^c) . \quad (27)$$

3.6 Bosonic equations

Using the results above, together with the conventions of Appendix A for the Dirac matrices and of Appendix B for the action of these matrices on the G_2 spinor, Eq. (5) leads to a set of bosonic first-order equations. They constitute our main formal result and are given by

$$J \cdot \varphi = 12K \cos \theta + \frac{1}{4}G \cdot \Phi \sin \theta \quad (28)$$

$$12V_A \cos \theta = G_{ABCD}\varphi^{BCD} + J^{BCD}\Phi_{ABCD} \sin \theta \quad (29)$$

$$\partial_y \alpha = \frac{1}{144}G \cdot \Phi \cos \theta - \frac{1}{3}K \sin \theta \quad (30)$$

$$\nabla_A \alpha = \frac{-1}{36}J^{BCD}\Phi_{ABCD} \cos \theta - \frac{1}{3}V_A \sin \theta \quad (31)$$

$$\partial_y v_\epsilon = \frac{1}{288} \left(e^{-i\theta} G \cdot \Phi - 8iJ \cdot \varphi - 48ie^{-i\theta}K \right) \quad (32)$$

$$\begin{aligned} \nabla_A \beta &= 2\partial_y (\text{Re}(v_A) \sin \theta + \text{Im}(v_A) \cos \theta) \\ &\quad + \frac{1}{18}J^{BCD}\Phi_{ABCD} \cos \theta + \frac{1}{6}V_A \sin \theta \end{aligned} \quad (33)$$

$$\partial_y \begin{pmatrix} \text{Re}(v_A) \cos \theta \\ +\text{Im}(v_A) \sin \theta \end{pmatrix} = \frac{-1}{72} (G_{ABCD}\varphi^{BCD} - 2J^{BCD}\Phi_{ABCD} \sin \theta + 6V_A \cos \theta) \quad (34)$$

$$\nabla_A v_\epsilon = \frac{-1}{72} \left(2iG_{ABCD}\varphi^{BCD} + e^{-i\theta}J^{BCD}\Phi_{ABCD} + 12ie^{-i\theta}V_A \right) \quad (35)$$

$$\begin{aligned} 4\nabla_A v_B + ie^{-i\theta}\partial_y h_{AB} - \nabla_C h_{AD}\varphi^{CD}_B &= \frac{1}{72} \left[\begin{aligned} &8iG_{ACDE}\Phi^{CDE}_B + \frac{i}{5}G^{CDEF}(4\Phi_{ACDE}\Omega_{FB} \\ &+ \Phi_{CDEF}\Omega_{AB} + 12\varphi_{ACD}\varphi_{EFB} + 8\varphi_{CDE}\varphi_{FAB}) \\ &- 24e^{-i\theta}J_{ACD}\varphi^{CD}_B - 4e^{-i\theta}J^{CDE}(3\varphi_{ACD}\Omega_{EB} - \\ &\varphi_{CDE}\Omega_{AB}) - 24ie^{-i\theta}V^C\varphi_{ABC} + 24K\Omega_{AB} \end{aligned} \right] . \end{aligned} \quad (36)$$

Here φ is the G_2 3-form on the internal space, as defined in Appendix B.1 and $\Phi = *\varphi$ is its 4-form dual.

These equations link the parameters (α, β, h) , associated with the metric, to the various flux components (G, J, V, K) and the quantities $(v_\epsilon, v^A, \theta)$ which parameterize the Killing spinor. We note that the y -derivative of β is unconstrained by these equations, which is as we would expect from a residual gauge degree of freedom after the choice of Ansatz. A solution to these first-order partial differential equations preserves two real supercharges ($N = 1/2$ from a four-dimensional point of view) and represents a warped product of a deformed G_2 space and a domain wall in four-dimensional space-time.

3.7 Ricci flatness

It is a general result [44, 45] that the integrability of the Killing spinor equation together with the field equations implies that the Einstein equations hold. To linear order in flux, this implies that our solutions should be Ricci-flat. Let us confirm this by explicitly computing the components of the Ricci tensor. Using the Ansatz (8) we find

$$\begin{aligned} R_{\mu\nu} &= (\partial_y^2 \alpha + \nabla_A^2 \alpha) \eta_{\mu\nu} \\ &= \left(\frac{-1}{144} (dJ - G') \cdot \Phi \cos \theta - 13(K' - d^*V) \sin \theta \right) \eta_{\mu\nu} \end{aligned} \quad (37)$$

$$\begin{aligned} R_{yy} &= 3\partial_y^2 \alpha + \nabla_A^2 \beta + \frac{1}{2} \partial_y^2 h_A^A \\ &= \frac{1}{72} (dJ - G') \cdot \Phi \cos \theta + \frac{1}{6} (K' - d^*V) \sin \theta \end{aligned} \quad (38)$$

$$\begin{aligned} R_{AB} &= \nabla^A \nabla_B (3\alpha + \beta) + \frac{1}{2} \partial_y^2 h_{AB} + \nabla_A \nabla_{[B} h_{C]}^C + \nabla^C \nabla_{[C} h_{B]A} - \frac{1}{2} h_D^C R^D_{ABC} \\ &= \frac{-1}{72} (dJ - G') \cdot \Phi \Omega_{AB} \cos \theta + \frac{1}{3} (K' - d^*V) \Omega_{AB} \sin \theta \\ &\quad + \frac{1}{6} (dJ - G')_{(A}{}^{CDE} \Phi_{B)CDE} \end{aligned} \quad (39)$$

$$\begin{aligned} R_{Ay} &= \partial_y (3\nabla_A \alpha + \nabla_{[A} h_{B]}) \\ &= -\frac{1}{72} (dG)_{BCDEF} \varphi_A{}^{BC} \varphi^{DEF} - \frac{1}{12} (dV)_{BC} \varphi^{BC}{}_A. \end{aligned} \quad (40)$$

With the first-order relations (28)–(36), together with the conditions on the flux (14) and (13), we find that these components of the Ricci tensor indeed vanish.

4 Explicit 11-dimensional solution

We now turn to the problem of integrating the bosonic equations (28)–(36). Since we are dealing with the case of a general G_2 manifold, this solution will take the form of a sum over basis sets of forms on the manifold. Although we have written the bosonic equations above in the ‘raw’ form in which they are obtained, there is a certain amount of hidden gauge symmetry that we would like to fix in our solution before we write down such an expansion. In this section, we also consider the zero-mode regime and its relation to compactification.

4.1 Simplifying the spinor Ansatz

Before we write down a solution to the 11-dimensional equations, we reconsider the spinor ansatz. Our 7-dimensional spinor should be invariant under $\text{SO}(7)$ transformations of the tangent space. Using the results of Appendix B.3 we can write such transformations as:

$$\chi \mapsto e^{\theta^{AB} \Sigma_{AB}} \chi = e^{\nu^A f_A + \mu^{AB} \rho_{AB}} \chi \quad (41)$$

where Σ_{AB} are taken in the spinor representation of $\text{SO}(7)$ and decompose into f_A, ρ_{AB} under G_2 and θ^{AB} , ν^A and μ^{AB} are real parameters. To first order, this transformation reads

$$\chi \mapsto (1 - v_0) \chi_0 + v^A \chi_A + \nu^A \chi_A \quad (42)$$

and means that we can ‘gauge away’ $\text{Re}(v^A)$. The effects of a general coordinate transformation on χ are similar but will in general yield weaker conditions on v^A .

A further point to note is that, since we have a Killing spinor, we are able to form bilinears in this spinor that will be globally defined. In particular, the global vector $W^I = \bar{\eta}\Gamma^I\eta$, formed in this way should itself be Killing [44]. At linear order, the transverse components of this vector are

$$W^y = \cos\theta + 2(\text{Im}(v_\epsilon)\sin\theta - \text{Re}(v_\epsilon)\cos\theta), \quad W^A = 4\text{Im}(v^A). \quad (43)$$

Since this vector must be Killing, we can then impose

$$\nabla_{(A}v_{B)} = 0, \quad \partial_y\text{Im}(v_A) = \frac{1}{2}\nabla_A(\text{Re}(v_\epsilon)\cos\theta - \text{Im}(v_\epsilon)\sin\theta). \quad (44)$$

These relations allow us to eliminate v^A from the Killing spinor equation for β which then takes the form

$$\nabla_A\beta = \frac{1}{36}G_{ABCD}\varphi^{BCD}\sin\theta\cos\theta + \frac{3}{72}J^{BCD}\Phi_{ABCD}\cos\theta + \frac{1}{6}V_A\sin\theta. \quad (45)$$

4.2 Simplifying the relations for metric perturbations

We also make a gauge choice for h_{AB} , by putting it in the standard ‘harmonic gauge’ so that

$$\nabla_B h^B_A = \frac{1}{2}\nabla_A\text{tr}(h). \quad (46)$$

Our result (28)–(36) can be simplified considerably by splitting into real and imaginary parts, projecting out the irreducible G_2 representations associated with the two free indices, using the simplifications of the spinor ansatz as above and making the harmonic gauge choice for h . We are then able to derive the following set of physically equivalent first order relations

$$J \cdot \varphi = 21K\cos\theta + \frac{5}{8}G \cdot \Phi\sin\theta \quad (47)$$

$$\partial_y\text{tr}(h) = -\frac{5}{72}G \cdot \Phi\cos\theta + \frac{7}{3}K\sin\theta \quad (48)$$

$$\nabla_A\text{tr}(h) = 4V_A\sin\theta - \frac{1}{3}J^{BCD}\Phi_{ABCD} \quad (49)$$

$$\partial_y(P_{27}h)_{AB} - \nabla_C(P_{27}h)_{D(A}\varphi^{CD}_{B)}\sin\theta = -\frac{1}{6}(P_{27}G)_{(A}{}^{CDE}\Phi_{B)CDE}\cos\theta \quad (50)$$

$$\begin{aligned} \nabla_C(P_{27}h)_{D(A}\varphi^{CD}_{B)}\cos\theta &= -\frac{1}{6}(P_{27}G)_{(A}{}^{CDE}\Phi_{B)CDE}\sin\theta \\ &\quad + \frac{1}{2}(P_{27}J)_{(A}{}^{CD}\varphi_{B)CD}, \end{aligned} \quad (51)$$

where the projector P_{27} projects out the **27** representation in the G_2 decomposition of the various tensors, as explained in Appendix B.2.

4.3 Zero-mode regime

The field equations (13) and (14) imply that

$$\begin{aligned} \Delta_7 G &= G'' & \Delta_7 J &= J'' \\ \Delta_7 V &= V'' & \Delta_7 K &= K'' \end{aligned} \quad (52)$$

where Δ_7 is the 7-dimensional Laplacian with respect to the zeroth order metric Ω . We call solutions for which both sides of these equations are zero the ‘zero modes’ and those for which both are equal to a non-zero constant the ‘massive modes’.

The physical reasoning behind this is that operators like Δ_7 will be associated with the inverse of the radius of compactification of X_7 . When this is reduced down to small scales, this makes Δ_7 produce extremely large constant non-zero eigenvalues, which are effective masses in the 4-dimensional theory. Since these masses will typically be at the Planck scale, they can be ignored in constructing the 4-dimensional effective theory, and so the zero-mode regime is of particular interest to us.

We firstly note that on G_2 manifolds there are no harmonic 1-forms, and so the following terms in the flux vanish in the zero-mode regime:

$$G_{ABCD}\varphi^{BCD} = J^{BCD}\Phi_{ABCD} = V_A = 0. \quad (53)$$

This also constrains the spinor so that $v^A = 0$, since otherwise the equation (36) would make v^A a harmonic 1-form. Such constraints on 7-dimensional vectors mean that in the zero-mode regime, using the first-order bosonic equations (28)–(36) we have

$$\nabla_A \alpha = \nabla_A \beta = 0. \quad (54)$$

A similar argument to that for the flux can be made for the graviton h_{AB} , which from (39), (46) and (54) obeys

$$\square_7 h = -\partial_y^2 h. \quad (55)$$

where

$$(\square_7 h)_{AB} := \nabla_C \nabla^C h_{AB} + 2R_{(AB)D}^C h_C^D. \quad (56)$$

In this case, we also argue that \square_7 will be associated with a Planck-scale effective mass upon compactification and so can be ignored.

The arguments above allow us to write a ‘zero-mode’ version of the first-order 11-dimensional bosonic equations (28)–(36)

$$\partial_y \alpha = \frac{1}{144} G \cdot \Phi \cos \theta - \frac{1}{3} K \sin \theta \quad (57)$$

$$\partial_y \text{tr}(h) = -\frac{5}{72} G \cdot \Phi \cos \theta + \frac{7}{3} K \sin \theta \quad (58)$$

$$\partial_y (P_{27} h)_{AB} = -\frac{1}{6} (P_{27} G)_{(A}{}^{CDE} \Phi_{B)CDE} \cos \theta \quad (59)$$

$$\partial_y \theta = -\frac{1}{48} G \cdot \Phi \sin \theta - K \cos \theta \quad (60)$$

$$P_{27} J = -P_{27} * G. \quad (61)$$

We shall use these equations later to compare with the bosonic equations that we derive from the 4-dimensional Killing spinor equations.

4.4 Fourier expansion of the flux

We shall now expand each component of the flux as a sum over forms on the G_2 manifold X_7 . At the zero-mode level, this expansion involves the harmonic forms and is given by

$$\begin{aligned} G_0 &= \sum_i G_i \psi^i, & V_0 &= 0 \\ J_0 &= \sum_i J_i * \psi^i, & K_0 &= \text{const.} \end{aligned} \quad (62)$$

Here G_i , J_i and K_0 are constants, and we have introduced a set of harmonic 4-forms on X_7 , $\{\psi^i\}_{i=1}^{b_3(X_7)}$ where $b_3(X_7)$ is the 3rd Betti number of X_7 . Notationally, we will sometimes adopt implicit summation over i, j type indices but leave them in for clarity at present.

In the massive regime, the expansion is slightly more complicated, since we must introduce a further set of massive 4-forms on X_7 , $\{\Psi^n\}$ satisfying

$$\Delta_7 \Psi^n = (m_n)^2 \Psi^n . \quad (63)$$

We can then use the Hodge star to construct a set of 3-forms, $\{*\Psi^n\}$, with

$$\Delta_7 * \Psi^n = (m_n)^2 * \Psi^n . \quad (64)$$

Then the massive modes of G, J can be expanded in terms of these forms, leading to

$$G_{\text{massive}} = \sum_n G_n(y) \Psi^n \quad J_{\text{massive}} = \sum_n J_n(y) * \Psi^n , \quad (65)$$

with y -dependent expansion coefficients G_n and J_n . The equations of motion for the flux then imply

$$\begin{aligned} G_n'' &= (m_n)^2 G_n \Rightarrow G_n(y) = G_n^+ e^{m_n y} + G_n^- e^{-m_n y} \\ J_n'' &= (m_n)^2 J_n \Rightarrow J_n(y) = J_n^+ e^{m_n y} + J_n^- e^{-m_n y} , \end{aligned} \quad (66)$$

for constant $G_n^+, G_n^-, J_n^+, J_n^-$. The massive expansion of K and V can be done in a similar way. We can write both in terms of a set of functions $\{f^p\}$ obeying $\Delta_7 f^p = (M_p)^2 f^p$ so that

$$\begin{aligned} V_{\text{massive}} &= \sum_p \frac{1}{M_p} (V_p^+ e^{M_p y} + V_p^- e^{-M_p y}) df^p \\ K_{\text{massive}} &= \sum_p (K_p^+ e^{M_p y} + K_p^- e^{-M_p y}) f^p \end{aligned} \quad (67)$$

for constants V_p^+, V_p^-, K_p^+ and K_p^- . We have introduced a factor of M_p in the first of these relations to compensate for the mass associated with the exterior derivative d . To see why this is the correct expansion for our solution consider the linear equation for $\nabla_A \alpha$, (31),

$$\begin{aligned} \nabla_A \alpha &= \frac{-1}{36} J^{BCD} \Phi_{ABCD} \cos \theta - \frac{1}{3} V_A \sin \theta = \\ (d\alpha)_A &= \frac{-1}{36} \sum_n (J_n^+ e^{m_n y} + J_n^- e^{-m_n y}) (*\Psi^n)^{BCD} \Phi_{ABCD} \cos \theta \\ &\quad - \frac{1}{3} \sum_p (V_p^+ e^{M_p y} + V_p^- e^{-M_p y}) (df^p)_A \sin \theta \end{aligned} \quad (68)$$

which, from the nilpotency of d implies that the right hand side of this equation is a closed 1-form and thus can be written uniquely as the sum of an exact 0-form and a harmonic 1-form. As there are no harmonic 1-forms on X_7 due to G_2 holonomy, each term on the right hand side must be exact. This justifies our expansion of V in terms of the $\{df^p\}$, and also allows us to define a further set of functions by

$$(d\Psi_{(0)}^n)_A := m_n (*\Psi^n)^{BCD} \Phi_{ABCD} . \quad (69)$$

4.5 Integration of the bosonic equations

After expanding as above, the direct integration of the bosonic equations (28)–(36) is relatively straightforward, particularly as we can take θ as a constant to linear order. Performing this integration then leads the following complete solution for the metric components

$$\alpha(y, x^A) = \left(\frac{-1}{3} K_0 y - \frac{1}{3} \sum_p \frac{1}{m_p} ((K_p^+ + V_p^+) e^{m_p y} - (K_p^- - V_p^-) e^{-m_p y}) f^p \right) \sin \theta =$$

$$\begin{aligned}
& + \frac{1}{144} \left(\sum_i G_i \psi^i y + \sum_n \frac{1}{m_n} (G_n^+ e^{m_n y} - G_n^- e^{-m_n y}) \Psi^n \right) \cdot \Phi \cos \theta \\
& - \frac{1}{36} \sum_n \frac{1}{m_n} (J_n^+ e^{m_n y} + J_n^- e^{-m_n y}) \Psi_{(0)}^n \cos \theta + \alpha_0
\end{aligned} \tag{70}$$

$$\begin{aligned}
\beta(y, x^A) &= \frac{1}{36} \left(\sum_i G_i \psi^i y + \sum_n \frac{1}{m_n} (G_n^+ e^{m_n y} - G_n^- e^{-m_n y}) \Psi^n \right)_{ABCD} \varphi^{BCD} \sin \theta \cos \theta \\
& + \frac{1}{6} \sum_p \frac{1}{m_p} (V_p^+ e^{m_p y} + V_p^- e^{-m_p y}) f^p \sin \theta \\
& + \frac{3}{72} \sum_n \frac{1}{m_n} (J_n^+ e^{m_n y} + J_n^- e^{-m_n y}) \Psi_{(0)}^n \cos \theta + \beta_0
\end{aligned} \tag{71}$$

$$\begin{aligned}
\text{tr}(h)(y, x^A) &= \left(\frac{7}{3} K_0 y + \sum_p \frac{1}{m_p} \left(\left(\frac{7}{3} K_p^+ + 4V_p^+ \right) e^{m_p y} - \left(\frac{7}{3} K_p^- - 4V_p^- \right) e^{-m_p y} \right) f^p \right) \sin \theta \\
& - \frac{5}{72} \left(\sum_i G_i \psi^i y + \sum_n \frac{1}{m_n} (G_n^+ e^{m_n y} - G_n^- e^{-m_n y}) \Psi^n \right) \cdot \Phi \cos \theta \\
& - \frac{1}{3} \sum_n \frac{1}{m_n} (J_n^+ e^{m_n y} + J_n^- e^{-m_n y}) \Psi_{(0)}^n \cos \theta + h_0 .
\end{aligned} \tag{72}$$

Here, α_0, β_0, h_0 are constants of integration and we have gauged away the y -dependence of β . Note that the first term, proportional to y on each right-hand side represents the zero-mode contributions while all other terms correspond to heavy modes. The above expressions make it clear that the massive modes are proportional to the inverse effective mass, and so upon compactification we expect them to be negligible. This means, we should consider only the zero mode regime when we come to the 4-dimensional effective theory later.

For the traceless part of the metric, the integration is slightly less straightforward, but proceeds along similar lines so that at zero mode

$$(P_{27} h_{\text{zero-mode}})_{AB} = -\frac{1}{6} \sum_i G_i y \cos \theta (P_{27} \psi^i)_{(A} {}^{CDE} \Phi_{B)CDE} . \tag{73}$$

The massive modes can be obtained by solving the equations

$$\begin{aligned}
& \partial_y (P_{27} h_{\text{massive}})_{AB} - \nabla_C (P_{27} h_{\text{massive}})_{D(A} \varphi_{B)}^{CD} \sin \theta = \\
& - \frac{1}{6} \sum_n (G_n^+ e^{m_n y} + G_n^- e^{-m_n y}) (P_{27} \Psi^n)_{(A} {}^{CDE} \Phi_{B)CDE} \cos \theta \\
& \nabla_C (P_{27} h_{\text{massive}})_{D(A} \varphi_{B)}^{CD} \cos \theta = \\
& - \frac{1}{6} \sum_n ((G_n^+ \sin \theta + J_n^+) e^{m_n y} + (G_n^- \sin \theta + J_n^-) e^{-m_n y}) (P_{27} \Psi^n)_{(A} {}^{CDE} \Phi_{B)CDE} ,
\end{aligned} \tag{74}$$

which can be done explicitly.

4.6 Curvature singularities

Analogously to the calculation in Ref. [30], we now look for the curvature singularities that correspond to vanishing volume of the compact space. This quantity is given by

$$V := \text{Vol}(X_7) = \int_{X_7} \sqrt{g_7} d^7 x . \tag{75}$$

We can then differentiate this to linear order using the relation (58) so that

$$\begin{aligned}
\partial_y V &= \int_{X_7} \frac{1}{2} \Omega^{AB} \partial_y h_{AB} \sqrt{\Omega} d^7 x \\
&= \int_{X_7} \left(-\frac{5}{144} G \cdot \Phi \cos \theta + \frac{7}{6} K \sin \theta \right) \sqrt{\Omega} d^7 x \\
&= \int_{X_7} \left(-\frac{5}{6} \varphi \wedge G \cos \theta + \frac{7}{6} * K \sin \theta \right) .
\end{aligned} \tag{76}$$

We can then use the equations of motion for the flux (13) and (14) to show that

$$\begin{aligned}
\partial_y^2 V &= \int_{X_7} d \left(-\frac{5}{6} (\varphi \wedge J) \cos \theta + \frac{7}{6} (*V) \sin \theta \right) \\
&= 0 ,
\end{aligned} \tag{77}$$

since X_7 has no boundary. The volume must thus depend linearly on the coordinate y , and so it must be zero for some value of y , which will correspond to a curvature singularity of the internal space. Of course, as the volume of the compact space becomes small, we are no longer entitled to use simply 11-dimensional supergravity as our theory. We might also reasonably expect that although the linear terms in flux in (77) vanish, the higher-order contributions may not.

5 Inclusion of brane sources

In general, we expect $(p+1)$ -form fields to be sourced by an extended charged p -brane. In M-theory, there are two sensible choices for p , given that the only form field present is the three-form field A . These are the ‘fundamental’ membrane (M2-brane) and the ‘magnetic’ five-brane (M5-brane).

We shall consider each of these in turn, as both may well support the kind of domain wall solution that we are considering. In the case of the M2-brane, this could happen by simply sitting in the external space, whereas the M5-brane would have to wrap a three-cycle in the compact space in an appropriate way.

Our approach shall be to solve the brane equations of motion for each system, and then try to match these solutions to appropriate specializations of the bulk solution that we have so far been considering. In doing this, we will find that the inclusion of a brane source fixes the value of θ in such a way that our bulk solution can either support the M2-brane or the M5-brane but not both.

We further find that in each case the brane splits the y -direction into two regions, each with different values of the flux and with a ‘jump’ in certain components of the flux across the brane proportional to the brane tension.

5.1 General brane action in M-theory

In this section, we will quote some general results about classical membranes in 11-dimensional supergravity, following Ref. [47]. The easiest to consider is the fundamental membrane, which couples to the 3-form field A by the action

$$S_2 = T_2 \int_{W_3} d^3 \mathcal{X} \left(-\frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^I \partial_j X^J g_{IJ} - \frac{1}{2} \sqrt{-\gamma} - \frac{1}{4!} \varepsilon^{ijk} \partial_i X^I \partial_j X^J \partial_k X^K A_{IJK} \right) , \tag{78}$$

where X^I are the brane coordinates, $i, j, \dots = 0, \dots, 2$ are brane worldvolume indices and W_3 is the worldvolume of the brane parameterized by the coordinates \mathcal{X}^i . This membrane couples to our

bulk SUGRA action S_{11} in Eq. (1) to produce the total action

$$S_{\text{Total}} = S_{11} + \int d^{11}x S_2 \delta(X - x) . \quad (79)$$

This modifies the previous equations of motion to give

$$dF = 0 \quad (80)$$

$$d^*F = -2\mathcal{J}_2 \quad (81)$$

$$R_{IJ} = \frac{1}{12} \left(F_{IKLM} F_J{}^{KLM} - \frac{1}{12} g_{IJ} F \cdot F \right) + \sqrt{-g} \left(T_{IJ} - \frac{1}{9} g_{IJ} T \right) . \quad (82)$$

The current and stress-energy associated with the brane are

$$\mathcal{J}_2^{IJK} = T_2 \int d^3\mathcal{X} \varepsilon^{ijk} \partial_i X^I \partial_j X^J \dots \partial_k X^K \frac{\delta^{11}(x - X)}{\sqrt{-g}} \quad (83)$$

$$T^{IJ} = -T_2 \int d^3\mathcal{X} \sqrt{-\gamma} \gamma^{ij} \partial_i X^I \partial_j X^J \frac{\delta^{11}(x - X)}{\sqrt{-g}} , \quad (84)$$

and the membrane worldvolume equations of motion are

$$\partial_i (\sqrt{-\gamma} \gamma^{ij} \partial_j X^J g_{IJ}) = \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^J \partial_j X^K \partial_I g_{JK} + \frac{1}{4!} \varepsilon^{ijk} \partial_i X^J \partial_j X^K \partial_k X^L F_{IJKL} \quad (85)$$

$$\gamma_{ij} = \partial_i X^I \partial_j X^J g_{IJ} . \quad (86)$$

The coupling of the five-brane to the bulk action is slightly more subtle, and to be done properly requires the approach of [48]. As we are only interested in the effect of the five-brane on the equations of motion rather than the duality-symmetric formulation of the low-energy M-theory action, we merely state here the five-brane equations of motion when we set the worldvolume two-form to zero. Analogously to the magnetic monopole in electrodynamics, the five-brane sources to the dual of the flux, $*F$, in the same way that the membrane couples to F . We then have bulk equations of motion

$$d^*F = 0 \quad (87)$$

$$dF = 2 * \mathcal{J}_5 \quad (88)$$

$$R_{IJ} = \frac{1}{12} \left(F_{IKLM} F_J{}^{KLM} - \frac{1}{12} g_{IJ} F \cdot F \right) + \sqrt{-g} \left(T_{IJ} - \frac{1}{9} g_{IJ} T \right) . \quad (89)$$

In this case the current and stress-energy are

$$\mathcal{J}_5^{I_1 \dots I_6} = T_5 \int d^6\mathcal{X} \varepsilon^{i_1 \dots i_6} \partial_{i_1} X^{I_6} \dots \partial_{i_6} X^{I_6} \frac{\delta^{11}(x - X)}{\sqrt{-g}} \quad (90)$$

$$T^{IJ} = -T_5 \int d^6\mathcal{X} \sqrt{-\gamma} \gamma^{ij} \partial_i X^I \partial_j X^J \frac{\delta^{11}(x - X)}{\sqrt{-g}} , \quad (91)$$

with worldvolume equations

$$\partial_i (\sqrt{-\gamma} \gamma^{ij} \partial_j X^J g_{IJ}) = \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^J \partial_j X^K \partial_I g_{JK} + \frac{1}{7!} \varepsilon^{i_1 \dots i_6} \partial_{i_1} X^{I_6} \dots \partial_{i_6} X^{I_6} (*F)_{I_1 \dots I_6} \quad (92)$$

$$\gamma_{ij} = \partial_i X^I \partial_j X^J g_{IJ} . \quad (93)$$

We shall now go on to apply these results for the M2-brane and M5-brane in the context of our existing bulk solution, noting that from the brane equations the flux is naturally of the order of the brane tension, which we must bear in mind when we truncate our results to linear order.

We will then go on to solve the membrane equations of motion at linear order, but only considering the zero-mode part, which will involve ‘smoothing over’ the 7-dimensional part of the delta functions in the action, essentially for reasons of simplicity. There is no clear reason why a fuller treatment of the massive modes should not yield fundamentally the same conclusions as below.

5.2 Fundamental membrane

It is natural to consider the membrane as supporting a domain wall solution by placing it in the external space with its transverse coordinate along the special direction, previously called y , of the solution. We implement this with the following ‘static gauge’ choice for the membrane coordinates

$$X^\mu = \xi^\mu, \quad X^y = \text{const.}, \quad X^A = \text{const.} \quad (94)$$

If we then consider the linearized 11-dimensional equations of motion, taking only the zero mode of the internal space part, we find the following modifications

$$\partial_y^2 \alpha = -\frac{2}{3} T_2 \delta(y) \quad (95)$$

$$\partial_y^2 \text{tr}(h) = \frac{14}{3} T_2 \delta(y) \quad (96)$$

$$K' = 2T_2 \delta(y). \quad (97)$$

The membrane worldvolume equations are

$$\partial_y \alpha = -\frac{1}{3} K \quad \nabla_A \alpha = -\frac{1}{3} V_A, \quad (98)$$

and imposing worldvolume supersymmetry (κ -symmetry) gives the following condition:

$$\widetilde{P}_- \eta = 0 \quad (99)$$

where $\widetilde{P}_\pm := \frac{1}{2}(1 \pm i\gamma^y \gamma) \otimes \mathbb{1}$ can be interpreted as projecting out different components of 8-D chirality. This condition turns out to be equivalent to taking $\sin \theta = 1$ in the Killing Spinor ansatz.

The physical interpretation of this solution is shown in Figure 1. Having smoothed over the compact space, we can consider the remaining 4-dimensional space to be split into two regions by the membrane, each containing different (constant) values for the flux.

Suppose we write the flux in Region I as K_I and in Region II as K_{II} , then we can integrate the equations (95)-(97) by writing

$$K(y) = 2T_2(K_{II} - K_I)\Theta(y) + K_I \quad (100)$$

$$\partial_y \alpha = -\frac{1}{3} K \quad \partial_y \text{tr}(h) = \frac{7}{3} K \quad (101)$$

where Θ is the step function defined by

$$\Theta(x) := \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases} \quad (102)$$

This solution is consistent with taking the $\sin \theta = 1$ specialization of the zero-mode bosonic equations (28)–(36) above in each of the bulk regions. The ‘jump’ in flux from one region to another is proportional to the membrane tension. It is worth noting that membranes could be stacked to make this jump proportional to the number of membranes times the tension.

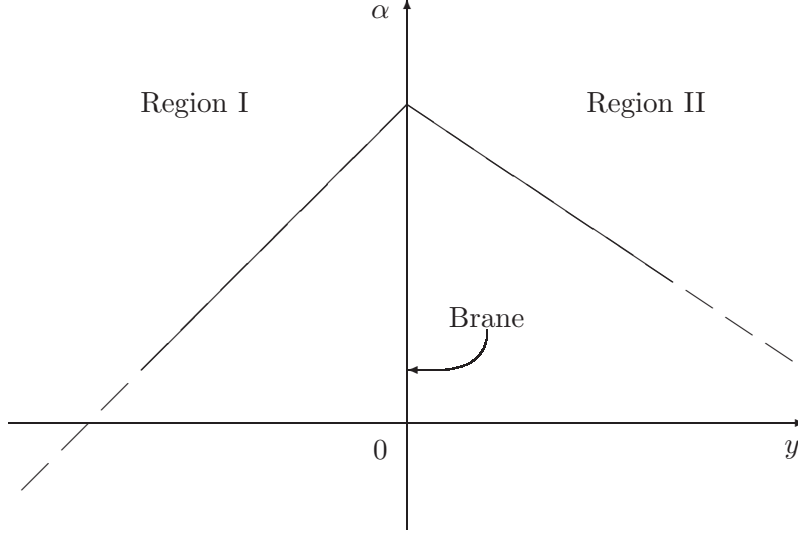


Figure 1: Physical Interpretation of the Membrane Solution

5.3 Magnetic five-brane

The situation for the M5-brane is slightly more complicated than for the M2-brane, given that it has three more worldvolume dimensions than the domain wall. For this reason, three of the worldvolume dimensions should be wrapped on some three-cycle Σ_3 within the G_2 space X_7 . For the solution that arises from this configuration to be supersymmetric, this cycle must be *calibrated* by the G_2 three-form φ , which means that

$$\text{Vol}(\Sigma_3) = \int_{\Sigma_3} \varphi . \quad (103)$$

The appropriate choice for the brane coordinates is then

$$X^\mu = \xi^\mu , \quad X^y = \text{const.} , \quad X^a = \sigma^a , \quad X^{\tilde{A}} = \text{const.} , \quad (104)$$

where we let σ^a be the coordinates of Σ_3 and use the indices \tilde{A} to denote directions perpendicular to Σ_3 . Considering the equations of motion for this system in the zero mode regime then gives

$$\partial_y^2 \alpha = \frac{-1}{3} T_5 \delta(y) \quad (105)$$

$$\partial_y^2 \text{tr}(h) = \frac{10}{3} T_5 \delta(y) \quad (106)$$

$$G' = \frac{2}{7} T_5 \delta(y) \Phi , \quad (107)$$

with worldvolume equations

$$6\partial_y \alpha + \frac{3}{7} \partial_y \text{tr}(h) = \frac{-1}{84} G \cdot \Phi \quad (108)$$

Imposing κ -symmetry gives the following condition:

$$P_-^y \eta = 0 \quad (109)$$

where $P_{\pm}^y := \frac{1}{2}(1 \pm \gamma^y) \otimes \mathbb{1}$ can be interpreted as projecting out different components of y -chirality. This condition turns out to be equivalent to taking $\cos \theta = 1$ in the Killing Spinor ansatz.

Our 4-dimensional picture then looks very similar to that of the membrane, with two separated regions containing different values for the flux, such that

$$G(y) = 2T_5(G_{\text{II}} - G_1)\Theta(y) + G_1. \quad (110)$$

Note that only the singlet part of G is shifted by the brane since, from Eq. (107), the difference $G_{\text{II}} - G_1$ is proportional to the G_2 invariant four-form Φ . As a result, we need not consider the traceless part of h and can thus simply integrate (105) and (106) to give

$$\partial_y \alpha = \frac{-1}{144} G \cdot \Phi, \quad \partial_y \text{tr}(h) = \frac{5}{72} G \cdot \Phi. \quad (111)$$

Similarly to the M2-brane case, this is consistent with the $\cos \theta = 1$ specialization of the bosonic equations (28)–(36). We also have the partition of the external space into two separate regions each with different constant values for the flux, with the jump in flux between these regions proportional to the brane tension. In contrast to the M2-brane, however, the relevant component of flux is the singlet of G rather than the K . It will also be possible to stack branes so that the jump in flux is proportional to the number of stacked branes.

Note that although both the membrane and five-brane very naturally couple to our bulk solution, at the order we are considering, the existence of supersymmetric configurations with two real supercharges containing both types of brane is ruled out.

6 Four-dimensional effective theory

When 11-dimensional SUGRA is compactified on a G_2 manifold, the effective field theory is given by 4-dimensional $N = 1$ SUGRA. In this section, we outline the action for this theory, together with how the quantities in that action are related to the 11-dimensional quantities. We then present the conditions for an $N=1$ supersymmetric domain wall solution in 4 dimensions, and integrate these equations. Finally, we uplift the 4-dimensional equations to 11 dimensions, and check that the result indeed matches our earlier one, obtained directly from the 11-dimensional theory.

6.1 For-dimensional $N = 1$ supergravity

The relevant bosonic terms in 4-dimensional $N = 1$ supergravity action are

$$S_4 = -\frac{1}{2} \int (\sqrt{-g} R + 2\mathcal{K}_{i\bar{j}} \partial_m T^i \partial^m \bar{T}^{\bar{j}} + 2U) \quad (112)$$

where $m = 0 \dots 3$. As we have done with its 11-dimensional counterpart, we have set the four-dimensional Newton constant κ_4 to one. The fields T^i are scalar components of chiral superfields, and $\mathcal{K}_{i\bar{j}}$ is the Kahler metric, given by

$$\mathcal{K}_{i\bar{j}} = \frac{\partial^2 \mathcal{K}}{\partial T^i \partial \bar{T}^{\bar{j}}} \quad (113)$$

in terms of the Kahler potential \mathcal{K} . Field indices i, j, \dots are lowered and raised by $\mathcal{K}_{i\bar{j}}$ and its inverse $\mathcal{K}^{i\bar{j}}$. The potential U is given in terms of the superpotential W by

$$U = e^{\mathcal{K}} (\mathcal{K}^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3|W|^2) \quad (114)$$

where the Kahler covariant derivative D_i is defined by

$$D_i := \frac{\partial}{\partial T^i} + \mathcal{K}_i. \quad (115)$$

6.2 4-dimensional SUGRA from M-Theory on a G_2 space

The relationship between the theory above and 11-dimensional SUGRA on a compact G_2 manifold is covered in Refs. [2] and [10]. Throughout this section we use just the zero-mode part of form field strength G which is written in terms of a set of harmonic 4-forms $\{\psi^i\}$ as

$$G = G_i \psi^i \quad (116)$$

where we now use implicit summation rather than the explicit notation of Section 4. Our first step is to expand both the 3-form field A and the G_2 3-form φ in terms of a dual set of harmonic three-forms $\{\pi_i\}_{i=1}^{b_3(X_7)}$ obeying

$$\int_{X_7} \pi_i \wedge \psi^j = \delta_i^j \quad (117)$$

so that

$$\varphi = \varphi^i \pi_i \quad (118)$$

$$A = A^i \pi_i + G_i \tau^i \quad (119)$$

where $d\tau^i = \psi^i$, so that $F = dA$ still holds. Detailed consideration of the compactification of the 11-dimensional theory shows that the φ^i are the metric moduli of the G_2 manifold, X_7 , and the moduli A^i appear as axions in the 4-dimensional theory. This means that we can write the superfields as

$$T^i = \varphi^i + iA^i. \quad (120)$$

In our conventions, the superpotential is then

$$W = \frac{7^{3/2}}{4} \int_{X_7} \left(\varphi + \frac{i}{2} A \right) \wedge G \quad (121)$$

$$= \frac{7^{3/2}}{4} G_i T^i. \quad (122)$$

We shall now consider the idea from Refs. [7, 10] that each term in (121) is ‘sourced’ by the wrapped M5-brane and M2-brane respectively. From (3)—the equation of motion for the flux—it can be shown that

$$\int_{X_7} \left(*K + \frac{1}{2} A \wedge G \right) = 0 \Rightarrow G_i A^i = -VK, \quad (123)$$

where V is the volume of the compact space as defined in (75). Also, as G is harmonic, the projector of its singlet must be constant over X_7 and so

$$G \cdot \Phi = \frac{1}{V} \int_{X_7} \sqrt{-g} G \cdot \Phi \Rightarrow G_i \varphi^i = \frac{1}{24} V G \cdot \Phi. \quad (124)$$

We can thus substitute (123) and (124) into (122) to rewrite the superpotential as

$$W = \frac{7^{3/2} V}{4} \left(\frac{1}{24} G \cdot \Phi - iK \right). \quad (125)$$

Clearly, this expression contains a term proportional to the singlet of G as well as one proportional to K . As we saw in section 5, the wrapped M5-brane acts as a source for the singlet of G and the M2-brane acts as a source for K . Therefore, we can interpret the moduli superpotential for

the 4-dimensional effective theory associated with M-theory on G_2 manifolds as being sourced by contributions from the wrapped M5-brane and the M2-brane.

The Kahler potential for this theory and its derivatives with respect to the superfields is given by

$$\mathcal{K} = -3 \ln(7V) \quad (126)$$

$$\Rightarrow \mathcal{K}_i = \frac{\partial \mathcal{K}}{\partial T^i} = \frac{-1}{2V} \int_{X_7} \pi_i \wedge \Phi = -2\varphi_i \quad (127)$$

$$\Rightarrow \mathcal{K}_{i\bar{j}} = \frac{1}{4V} \int_{X_7} \pi_i \wedge * \pi_{\bar{j}} . \quad (128)$$

Using these expressions, we can write G in terms of the dual set of 4-forms like

$$G = \frac{1}{4V} G^i * \pi_i . \quad (129)$$

This is all the input from 11 dimensions that we need to write down and solve the appropriate 4-dimensional Killing spinor equations. We will return to the links between 4 and 11 dimensions later.

6.3 4-dimensional Killing spinor equations

We shall now set up the conditions for supersymmetric domain wall solutions in the 4-dimensional supergravities derived from our 11-dimensional theory as above. We start with a warped metric ansatz

$$ds_4^2 = e^{a(z)} \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \quad (130)$$

and z -dependent scalar fields $T^i = T^i(z)$. For such a field configuration the first order relations¹ derived from the Killing spinor equations are given by [46]

$$\partial_z a = e^{-i\theta} e^{\frac{1}{2}\mathcal{K}} W \quad (131)$$

$$\partial_z T^i = -e^{i\theta} e^{\frac{1}{2}\mathcal{K}} \mathcal{K}^{i\bar{j}} D_{\bar{j}} \bar{W} \quad (132)$$

$$\partial_z \theta = -\text{Im} [(\partial_z T^i) \mathcal{K}_i] . \quad (133)$$

Here, θ parameterizes the 4-dimensional Killing spinor in a similar way to the quantity θ we have used to parameterize the earlier 11-dimensional spinor. When we consider the links between 4 and 11 dimensions later, we will find that they are in fact the same at zero mode level in 11 dimensions.

Solutions to these equations are BPS states which preserve half of the maximum number of supercharges possible in the $N = 1$, $D = 4$ theory, and automatically satisfy its equations of motion. Using the expressions for T^i , W and \mathcal{K} from 11 dimensions above allows us to write the first-order relations as

$$\partial_z a = \frac{V^{-3/2}}{4} (G_i \varphi^i \cos \theta + G_i A^i \sin \theta) \quad (134)$$

$$G_i A^i \cos \theta = G_i \varphi^i \sin \theta \quad (135)$$

$$\partial_z \varphi^i = \frac{V^{-3/2}}{4} ((2G_j \varphi^j \varphi^i - G^i) \cos \theta + 2G_j A^j \varphi^i \sin \theta) \quad (136)$$

¹There is one further equation that constrains the 4-dimensional Killing spinor but it does not really impact on our calculation.

$$\partial_z A^i = \frac{V^{-3/2}}{4} ((2G_j \varphi^j \varphi^i - G^i) \sin \theta - 2G_j A^j \varphi^i \cos \theta) \quad (137)$$

$$\partial_z \theta = -\frac{V^{-3/2}}{2} G_i \varphi^i \sin \theta . \quad (138)$$

It is also worth noting that the relation

$$\partial_z V = \frac{V^{-1/2}}{6} (5G_i \varphi^i \cos \theta + 7G_i A^i \sin \theta) . \quad (139)$$

can be derived from Eqs. (134)–(138). This will be important later when we find explicit solutions.

We now consider how to integrate these 4-dimensional equations in purely 4-dimensional language, before uplifting them to compare with the 11-dimensional equations.

6.4 Integrating the 4-dimensional equations

We now present the most general solution to equations (131)–(133), which are the conditions for a supersymmetric domain wall to 4-dimensional SUGRA, given that the SUGRA descends from M-theory on a G_2 manifold. The solution can be written, in terms of new moduli fields f^i , as

$$a = \frac{1}{2} \ln |\cot \theta| + C \quad (140)$$

$$\varphi_i = \tan(\theta) f_i \quad (141)$$

$$A^i = \frac{-1}{4V_f \cot^{-7/3} \theta} G^i u + b^i , \quad (142)$$

where the new transverse coordinate u is related to z by

$$\partial_u = (V_f^{1/2} \cot^{1/2} \theta \operatorname{cosec} \theta) \partial_z . \quad (143)$$

Here, C and b^i are constants of integration. Recall, that once the Kahler potential is explicitly given, the volume V is known, via Eq. (126), as a function of the moduli φ^i . By V_f we mean this function but with the fields φ^i replaced by f^i . Since the volume is a homogeneous function of degree $7/3$ in its arguments this means V and V_f are related by

$$V_f = V \cot^{7/3} \theta . \quad (144)$$

The angle θ is fixed by

$$\cos \theta \cot^6 \theta = \left(\frac{V_f}{V_0} \right)^{3/2} , \quad (145)$$

where V_0 is another constant of integration. Finally, the new field f_i are linear functions

$$f_i = \frac{1}{4} G_i w + k_i , \quad (146)$$

in the transverse coordinate w defined by

$$\partial_w = (V_f^{3/2} \tan^{9/2} \theta \sec \theta) \partial_z , \quad (147)$$

and k_i are further constants of integration. This completes the most general domain wall solution to 4-dimensional $N = 1$ supergravity theories from M-theory flux compactifications on G_2 spaces.

Note that the above solutions display some apparent singularities in the cases $\sin \theta = 1$ or $\cos \theta = 1$ which we have previously encountered when matching to membrane and five-brane sources. However, these singularities are not real but can be removed by introducing a new quantity n defined by either one of the two equivalent relations

$$n = \frac{3\ln|\cot \theta|}{7\ln|\cos \theta| - 5\ln|\sin \theta|} , \quad \left(\frac{V}{V_0}\right)^n = |\cot \theta| . \quad (148)$$

In the above solution, we can in general eliminate θ in favour of n .

In this new form one can explicitly consider the cases $\sin \theta = 1$ and $\cos \theta = 1$ without encountering any singularities. They lead to the particularly simple solutions

$\cos \theta = 1$	$\sin \theta = 1$
$\partial_w = V^{9/10} \partial_z$	$\partial_u = V^{1/2} \partial_z$
$f_i = V^{3/5} \varphi_i = \frac{1}{4v} G_i w + k_i$	$f_i = V^{3/7} \varphi_i = \text{const.}$
$A^i = \text{const.}$	$A^i = -\frac{1}{4V} G^i u + b^i$
$a = a_0 + \ln V^{3/10}$	$a = a_0 + \ln V^{3/14}$

(149)

The results in this section represent the most general supersymmetric domain wall solution to 4-dimensional supergravities that arise from compactification of M-theory on G_2 manifolds with flux. In particular, the solution with $\cos \theta = 1$ in the above table is the appropriate one to match to a wrapped 5-brane source while the solution for $\sin \theta = 1$ can be matched to a membrane source. We note the linear dependence of the moduli fields on the natural transverse coordinate w or u , which will have the effect of causing fields to diverge at large distances. In particular, the solutions do not approach four-dimensional Minkowski space asymptotically. This behaviour is typical for supergravity domain walls supported by potentials without a minimum at any finite field value [49]. We will discuss curvature singularities in section 6.6 below.

6.5 Comparison with 11 dimensions

As a check of our results, we shall now test for compatibility between the 4- and 11-dimensional relations derived from the Killing spinor equations. Our strategy will be to link the 4- and 11-dimensional quantities and then uplift the 4-dimensional equations to 11 dimensions, and see if they agree.

We firstly relate the quantities. By comparing the 4-dimensional and 11-dimensional line elements ds_4^2 and ds_{11}^2 we see that

$$\partial_y = \pm \sqrt{V} \partial_z \quad (150)$$

$$\alpha = a - \frac{1}{2} \ln(V) . \quad (151)$$

Using the result (217) of Appendix B.4 it is clear that to first order

$$\partial_y h_{AB} = \frac{1}{2} (P_{27} \partial_y \varphi)_{(A} {}^{CD} \varphi_{B)CD} + \frac{1}{63} (\partial_y \varphi) \cdot \varphi \Omega_{AB} . \quad (152)$$

Making use of relations (124) and (123), we simply substitute these results into the 4-dimensional bosonic equations for appropriate sign choice in (150) and get the following

$$\partial_y \alpha = \frac{1}{144} G \cdot \Phi \cos \theta - \frac{1}{3} K \sin \theta \quad (153)$$

$$\partial_y \text{tr}(h) = -\frac{5}{72} G \cdot \Phi \cos \theta + \frac{7}{3} K \sin \theta \quad (154)$$

$$\partial_y (P_{27} h)_{AB} = -\frac{1}{6} (P_{27} G)_{(A}{}^{CDE} \Phi_{B)CDE} \cos \theta \quad (155)$$

$$\partial_y \theta = -\frac{1}{48} G \cdot \Phi \sin \theta - K \cos \theta \quad (156)$$

$$G \cdot \Phi = 24K \cos \theta \quad (157)$$

$$J \cdot \varphi = 21K \cos \theta + \frac{5}{8} G \cdot \Phi \sin \theta \quad (158)$$

Although the equations linking components of the flux are superficially different from the 11-dimensional ones, they are easily shown to be equivalent.

We note here that while the 4-dimensional equations did not involve linearization in the flux while the 11-dimensional ones did. Nevertheless, it turns out by comparing Eqs. (153)–(158) with Eqs. (57)–(61) that the two sets of equations are equivalent at the zero mode level.

6.6 Curvature singularities

Another feature of the 4-dimensional equations that we should compare with 11-dimensions is the variation of the volume as a function of the transverse coordinate. A zero of this volume at any particular point in the transverse space implies, of course, a curvature singularity of the internal space. However, it can be shown that such a vanishing internal volume also leads to a four-dimensional curvature singularity.

The z -variation of the volume is described by Eq. (139) which uplifts to give the following

$$\partial_y V = \left[-\frac{5}{144} G \cdot \Phi \cos \theta + \frac{7}{6} K \sin \theta \right] V. \quad (159)$$

This is the generalization to all orders in flux of the 11-dimensional result (76) in the zero-mode regime. When we consider the second derivative of Eq. (159), using our 4-dimensional equations, we get

$$\partial_y^2 V = \frac{-1}{12V} \left[(G_i \varphi^i)^2 (2 \sin^2 \theta - 5) + \frac{1}{2} G_i G^i (2 \sin^2 \theta + 5) \right], \quad (160)$$

which is quadratic in the flux. Writing this in 11-dimensional language gives

$$\partial_y^2 V = \frac{-1}{12V} (2 \sin^2 \theta - 5) \left[\int_{X_7} \varphi \wedge G \right]^2 - \frac{1}{6} (2 \sin^2 \theta + 5) \int_{X_7} *G \wedge G. \quad (161)$$

This vanishes at linear order in flux which is consistent with our earlier findings in Eq. (77). These implied a linear behaviour of the volume and, therefore, its vanishing at some finite coordinate y . However, the above result, good to all orders in flux, shows that, in fact, the volume varies in a more complicated way. In particular, the vanishing of the volume at some finite y cannot be deduced generically at this stage. To decide whether or not the volume vanishes one may study specific examples of G_2 manifolds where V is given as an explicit function of the fields, as in Ref. [19].

7 Conclusions

In this paper, we have studied M-theory compactifications on G_2 spaces in the presence of flux, both from the viewpoint of the 11-dimensional theory and the associated 4-dimensional supergravity theories. We have solved the 11-dimensional Killing spinor equations to linear order in flux

and obtained G_2 domain walls, consisting of a warped product of a deformed G_2 space and a domain wall in four-dimensional space-time. The zero mode parts of these solutions have also been reproduced, to all orders in flux, by solving the Killing spinor equations of the associated effective four-dimensional $N = 1$ supergravity theories, obtained by reducing M-theory on G_2 spaces with flux. From this four-dimensional perspective, the solutions are domain walls which couple to the flux superpotential and with moduli varying non-trivially along the transverse direction. This transverse variation of the scalar fields can be seen as a path in the moduli space of G_2 metrics or, in other words, a variation of the internal G_2 space as one goes along the transverse direction.

We have also shown that these domain wall solutions can be sourced by either a membrane in four-dimensional space-time or an M5-brane wrapping a 3-cycle within the G_2 space. This leads to an interpretation of our solutions as the simplest manifestation of an M-theory “topological defect” membrane or wrapped 5-brane appearing in “our” four-dimensional universe. We believe that studying such defects, arising from wrapped branes, in the context of M-theory cosmology is an interesting problem.

Our four-dimensional domain walls diverge away from the wall and, in particular, do not approach Minkowski space. This is a common feature of supergravity domain walls [49]. It would be interesting to see how these solutions are modified by the inclusion of a non-perturbative superpotential in four dimensions and whether this can remove the divergences. We hope to return to this question in a future publication.

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A Spinor conventions

Throughout this paper we have made use of certain properties of the Dirac matrices in various dimensions. We collect these here for reference.

A.1 Three dimensions

In our conventions, the 3-dimensional Minkowski Dirac matrices are given by

$$\begin{aligned}\rho^0 &= -i\sigma^2 \\ \rho^1 &= \sigma^1 \\ \rho^2 &= -\sigma^3 .\end{aligned}\tag{162}$$

These obey the following

$$\rho^0 \rho^1 \rho^2 = -\mathbb{1}_2\tag{163}$$

$$(\rho^\mu)^* = \rho^\mu\tag{164}$$

$$\{\rho^\mu, \rho^\nu\} = 2\eta^{\mu\nu}\tag{165}$$

where $\mu, \nu = 0, 1, 2$.

A.2 Four dimensions

The 4-dimensional Dirac matrices are constructed from the above by

$$\begin{aligned}\gamma^\mu &= \rho^\mu \otimes \sigma^1 \\ \gamma^y &= \mathbb{1}_2 \otimes \sigma^2.\end{aligned}\tag{166}$$

We can define a 4D chirality operator $\gamma = i\gamma^0\gamma^1\gamma^2\gamma^y$ so that

$$\{\gamma, \gamma^y\} = \{\gamma, \gamma^\mu\} = \{\gamma^\mu, \gamma^y\} = 0\tag{167}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\tag{168}$$

$$(\gamma)^2 = (\gamma^y)^2 = 1.\tag{169}$$

These matrices further obey the following

$$\hat{\varepsilon}_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho = 6i\gamma^y\gamma\tag{170}$$

$$\hat{\varepsilon}_{\nu\rho}^\mu\gamma^\nu\gamma^\rho = 2i\gamma^y\gamma\gamma^\mu\tag{171}$$

$$(\gamma^\mu)^* = \gamma^\mu \quad (\gamma)^* = \gamma \quad (\gamma^y)^* = -\gamma^y \quad ,\tag{172}$$

where $\hat{\varepsilon}_{012} := 1$.

A.3 Seven dimensions

In 7-dimensional Euclidean space it is possible to define a set of Dirac matrices $\{\Upsilon^A\}_{A=4\dots 10}$ that are purely imaginary so that

$$(\Upsilon^A)^* = -\Upsilon^A\tag{173}$$

$$\{\Upsilon^A, \Upsilon^B\} = 2\delta^{AB}.\tag{174}$$

A.4 11 dimensions

Finally, we define the set of 11-dimensional Dirac matrices

$$\{\Gamma^I\}_{I=0\dots 10} = \{\Gamma^\mu, \Gamma^y, \Gamma^A\}_{\mu=0,1,2, A=4\dots 10}\tag{175}$$

by the relations

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes \mathbb{1}_8 \\ \Gamma^y &= \gamma^y \otimes \mathbb{1}_8 \\ \Gamma^A &= \gamma \otimes \Upsilon^A.\end{aligned}\tag{176}$$

Throughout this paper we use the standard notation

$$\Gamma^{I_1\dots I_p} = \Gamma^{[I_1\dots I_p]},\tag{177}$$

for anti-symmetrized products of Dirac matrices.

A.5 Majorana Conjugation

Since 11-dimensional supergravity is parameterized by Majorana spinors, it is useful to be clear about our conventions for charge conjugation and, in view of our compactification, how this operation decomposes under a product spinor Ansatz.

For a general spinor ψ , we define its Majorana conjugate in terms of the matrix B

$$\psi^c = B^{-1}\psi^* . \quad (178)$$

Imposing that this operation should commute with Lorentz transformations and square to unity gives the conditions

$$B\Gamma^I B^{-1} = \pm(\Gamma^I)^* \quad B^* B = \mathbb{1} . \quad (179)$$

Let us decompose B as $B^{-1} = B_4^{-1} \otimes B_7^{-1}$ into 4- and a 7-dimensional conjugation matrices B_4 and B_7 . They must satisfy the relations

$$B_4\gamma^m B_4^{-1} = \pm(\gamma^m)^* \quad B_7\Upsilon^A B_7^{-1} = \mp(\Upsilon^A)^* \quad (180)$$

where $m = 0, 1, 2, y$, in order to reproduce Eq. (179). In our conventions, these matrices can be represented as

$$B_4^{-1} = \gamma^y \gamma \quad B_7^{-1} = \mathbb{1}_8 . \quad (181)$$

B Properties of G_2

We shall now state some results for manifolds of G_2 holonomy, in part following Ref. [50, 8].

B.1 G_2 Lie Algebra

In this section we define G_2 as a subgroup of $\text{SO}(7)$ by defining the G_2 3-form φ . We also decompose the Lie algebra of $\text{SO}(7)$ into a part in G_2 and a perpendicular part.

$\text{SO}(7)$ and its Lie algebra are given by

$$\text{SO}(7) := \{O \in \text{Gl}(7, \mathbb{R}) \mid O = O^T \text{ \& \; } \det(O) = 1\} \quad (182)$$

$$\begin{aligned} \Rightarrow \mathcal{L}(\text{SO}(7)) &= \{T \mid T^T = -T\} \\ &= \text{Span}\{S_{AB}\} . \end{aligned} \quad (183)$$

where the basis of 21 generator matrices is

$$(S_{AB})^C_D := \delta_A^C \delta_{BD} - \delta_{AD} \delta_B^C . \quad (184)$$

G_2 is then defined as the subgroup of $\text{SO}(7)$ whose action preserves the 3-form φ

$$\varphi = d\mathbf{x}^{123} + d\mathbf{x}^{145} + d\mathbf{x}^{167} + d\mathbf{x}^{246} - d\mathbf{x}^{257} - d\mathbf{x}^{347} - d\mathbf{x}^{356} \quad (185)$$

where $d\mathbf{x}^{A_1 \dots A_p} := dx^{A_1} \wedge \dots \wedge dx^{A_p}$. G_2 also preserves the Hodge dual of this 3-form, $\Phi := *\varphi$. We may thus write

$$G_2 := \left\{ P \in \text{SO}(7) \mid P_A^{A'} P_B^{B'} P_C^{C'} \varphi_{A'B'C'} = \varphi_{ABC} \right\} \quad (186)$$

$$\Rightarrow \mathcal{L}(G_2) = \left\{ T \in \mathcal{L}(\text{SO}(7)) \mid T_A^{A'} \varphi_{A'BC} + T_B^{B'} \varphi_{AB'C} T_C^{C'} \varphi_{ABC'} = 0 \right\} . \quad (187)$$

We can then split the $\text{SO}(7)$ generators into a group of 7 not in $\mathcal{L}(G_2)$ and a group of 14 in $\mathcal{L}(G_2)$, which are given by

$$F_A = \frac{1}{2}\varphi_A{}^{BC}S_{BC} \quad (188)$$

$$R_{AB} = \frac{2}{3}S_{AB} - \frac{1}{6}\Phi_{AB}{}^{CD}S_{CD} \quad (189)$$

respectively. This represents the branching rule

$$\mathbf{21}_{\text{SO}(7)} \rightarrow (\mathbf{14} + \mathbf{7})_{G_2} . \quad (190)$$

B.2 Projector conventions

In solving equations containing G_2 spinors, vectors, and tensors, it is usually easiest to project out irreducible representations of the relevant indices. Here we summarize our conventions for the projectors that we use.

For a 2-form, ξ , decomposing as $(\mathbf{7} \times \mathbf{7})_{\text{anti-symm.}} \rightarrow \mathbf{7} + \mathbf{14}$, we have

$$(P_7\xi)_{AB} = \frac{1}{6}(\xi_{CD}\varphi^{CDE}\varphi_{EAB}) \quad (191)$$

$$P_{14} = \mathbb{1} - P_7 . \quad (192)$$

For a 3-form, ζ , decomposing as $(\mathbf{7} \times \mathbf{7} \times \mathbf{7})_{\text{anti-symm.}} \rightarrow \mathbf{1} + \mathbf{7} + \mathbf{27}$ we have

$$(P_1\zeta)_{ABC} = \frac{1}{42}(\varphi \cdot \zeta)\varphi_{ABC} \quad (193)$$

$$(P_7\zeta)_{ABC} = \frac{1}{24}\Phi^{DEFG}\zeta_{EFG}\Phi_{DABC} \quad (194)$$

$$P_{27} = \mathbb{1} - P_1 - P_7 . \quad (195)$$

For a 4-form, χ , decomposing as $(\mathbf{7} \times \mathbf{7} \times \mathbf{7} \times \mathbf{7})_{\text{anti-symm.}} \rightarrow \mathbf{1} + \mathbf{7} + \mathbf{27}$ we have

$$(P_1\chi)_{ABCD} = \frac{1}{168}(\Phi \cdot \chi)\Phi_{ABCD} \quad (196)$$

$$(P_7\chi)_{ABCD} = \frac{1}{42}\varphi^{EFG}\chi_{EFG[A}\varphi_{BCD]} \quad (197)$$

$$P_{27} = \mathbb{1} - P_1 - P_7 . \quad (198)$$

For a symmetric rank 2 tensor, S , decomposing as $(\mathbf{7} \times \mathbf{7})_{\text{symmetric}} \rightarrow \mathbf{1} + \mathbf{27}$, we have

$$(P_1S)_{AB} = \frac{1}{7}S^C{}_C\Omega_{AB} \quad (199)$$

$$P_{27} = \mathbb{1} - P_1 . \quad (200)$$

For an $\text{SO}(7)$ spinor, decomposing under G_2 as $\mathbf{8}_{\text{SO}(7)} \rightarrow (\mathbf{1} + \mathbf{7})_{G_2}$ we have

$$P_1 = \frac{1}{8}\left(\mathbb{1} - \frac{1}{4!}\Phi_{ABCD}\Upsilon^{ABCD}\right) \quad (201)$$

$$P_7 = \mathbb{1} - P_1 . \quad (202)$$

B.3 G_2 spinors

In this section, we construct a basis set for general spinors on a G_2 manifold using some of the results from earlier appendices. We also consider the action of our Dirac matrices on these spinors, which is crucial in solving the Killing spinor equations.

The 7-dimensional Dirac matrices can be used to provide a representation of $\mathcal{L}(\text{SO}(7))$ via

$$\Sigma_{AB} := \frac{1}{2} \Upsilon_{AB} \quad (203)$$

which obey the same commutation relations as the S_{AB} above. We can use this isomorphism to split the Σ_{AB} as for Appendix B.1

$$f_A = \frac{1}{2} \varphi_A{}^{BC} \Sigma_{BC} \quad (204)$$

$$\rho_{AB} = \frac{2}{3} \Sigma_{AB} - \frac{1}{6} \Phi_{AB}{}^{CD} \Sigma_{CD} . \quad (205)$$

We are then able to define a ‘zeroth order’ spinor χ_0 by

$$P_1 \chi_0 = \chi_0 \quad P_7 \chi_0 = 0 \quad \overline{\chi_0} \chi_0 = 1 \quad (206)$$

which is covariantly conserved on the G_2 manifold, i.e. $\rho_{AB} \chi_0 = 0$. We further define a set of seven other spinors $\{\chi_A\}$ obeying

$$\begin{aligned} \chi_A &:= \frac{2}{3} f_A \chi_0 & P_7 \chi_A &= \chi_A \\ \overline{\chi_0} \chi_A &= \overline{\chi_A} \chi_0 = 0 & \overline{\chi_A} \chi_B &= \delta_{AB} \end{aligned} \quad (207)$$

which represents the branching rule

$$\mathbf{8}_{\text{SO}(7)} \rightarrow (\mathbf{7} + \mathbf{1})_{G_2} . \quad (208)$$

We therefore have $\{\chi_0, \chi_A\}$ as a basis for spinors on a G_2 manifold. The action of our Dirac matrices on these spinors is then given by

$$\Upsilon^A \chi_0 = i \Omega^{AB} \chi_B \quad (209)$$

$$\Upsilon^A \chi_B = -i \delta_B^A \chi_0 + i \varphi_B{}^{AC} \chi_C \quad (210)$$

$$\Upsilon^{AB} \chi_0 = \varphi^{ABC} \chi_C \quad (211)$$

$$\Upsilon^{AB} \chi_C = -\varphi_C{}^{AB} \chi_0 + (\Phi_C{}^{ABD} + \delta_C^B \Omega^{AD} - \delta_C^A \Omega^{BD}) \chi_D \quad (212)$$

$$\Upsilon^{ABC} \chi_0 = -i \varphi^{ABC} \chi_0 + i \Phi^{ABCD} \chi_D \quad (213)$$

$$\Upsilon^{ABCD} \chi_0 = -\Phi^{ABCD} \chi_0 - 4 \varphi^{[ABC} \Omega^{D]E} \chi_E \quad (214)$$

$$\Upsilon^{ABCDE} \chi_0 = -i \left(\Phi^{[ABCD} \Omega^{E]F} + 4 \varphi^{[ABC} \varphi^{DE]F} \chi_F \right) \quad (215)$$

where φ is the G_2 3-form, Φ is its dual, Ω is the metric on the G_2 space and the Υ are the 7-dimensional Dirac matrices.

B.4 G_2 induced metric

The G_2 form φ naturally induces a metric on a G_2 manifold

$$\begin{aligned}
\Omega_{AB} &= (\det(s))^{-1/9} s_{AB} \text{ for} \\
s_{AB} &= \frac{1}{144} \varphi_{ACD} \varphi_{BEF} \varphi_{GHI} \hat{\epsilon}^{CDEFGHI}
\end{aligned} \tag{216}$$

where the entries of the completely antisymmetric tensor $\hat{\epsilon}$ are either 1 or 0. By writing $\varphi = \varphi^i \pi_i$ it is then possible to show that

$$\partial g_{AB} = \left[\frac{1}{2} \varphi_{(A}{}^{CD} (\pi_i)_{B)CD} - \frac{1}{18} (\varphi \cdot \pi_i) g_{AB} \right] \partial \varphi^i \tag{217}$$

using some of the identities below.

B.5 G_2 identities

Throughout this paper, the following G_2 identities have been used:

$$\varphi \cdot \varphi = 42 \tag{218}$$

$$\varphi^{ACD} \varphi_{BCD} = 6 \delta_B^A \tag{219}$$

$$\varphi_{ABE} \varphi_{CD}{}^E = \Phi_{ABCD} + \Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC} \tag{220}$$

$$\Phi \cdot \Phi = 168 \tag{221}$$

$$\Phi^{ACDE} \Phi_{BCDE} = 24 \delta_B^A \tag{222}$$

$$\Phi_{ABEF} \Phi_{CD}{}^{EF} = 2 \Phi_{ABCD} + 4(\Omega_{AC} \Omega_{BD} - \Omega_{AD} \Omega_{BC}) \tag{223}$$

$$\varphi^{BCD} \Phi_{ABCD} = 0 \tag{224}$$

$$\varphi_A{}^{DE} \Phi_{BCDE} = 4 \varphi_{ABC} \tag{225}$$

$$\begin{aligned}
\varphi_{AB}{}^F \Phi_{FCDE} &= \Omega_{AC} \varphi_{BDE} + \Omega_{AD} \varphi_{BEC} + \Omega_{AE} \varphi_{BCD} - \\
&\quad \Omega_{BC} \varphi_{ADE} - \Omega_{BD} \varphi_{AEC} - \Omega_{BE} \varphi_{ACD}
\end{aligned} \tag{226}$$

$$\varphi_{[ABC} \Phi_{DEFG]} = \frac{1}{5} \epsilon_{ABCDEFGH} \tag{227}$$

$$3 \varphi_{(A}{}^{[CD} \varphi_{B)}{}^{EF]} + 4 \Phi^{CDE}{}_{(A} \delta_{B)}^F - \Phi^{CDEF} \Omega_{AB} = 0 \tag{228}$$

$$\epsilon^{CDEFGHI} \varphi_{ACD} \varphi_{BEF} = 24 \varphi^{[GH}{}_{(A} \delta_{B)}^I] , \tag{229}$$

where φ is the G_2 3-form, Φ is its dual and Ω is the metric on the G_2 space. For a 4-form α such that $P_7 \alpha = 0$, we have the further results that

$$\alpha \cdot \Phi = 4(*\alpha) \cdot \varphi \tag{230}$$

$$(P_{27} \alpha)_{(A}{}^{CDE} \Phi_{B)CDE} = -3(P_{27} * \alpha)_{(A}{}^{CD} \varphi_{B)CD} . \tag{231}$$

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